IMU Propagation as Preintegration

paper id:419

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isprs 2026
TORONTO
XXV ISPRS CONGRESS
TORONTO, CANADA | 4-11 JULY 2026
FROM IMAGERY TO UNDERSTANDING

Propagation Equivalent to Preintegration?

Research questions:

1. Can existing IMU propagation code be reused for preintegration?
2. To what extent can an existing preintegration implementation be reused when the error-state definition changes?
3. Can IMU propagation be used to validate preintegration, especially its covariance estimation?

Propagation and Preintegration

Propagation Basics

Given 6-axis IMU measurements $\mathbf{z}_k = [\boldsymbol{\omega}_k^m, \mathbf{a}_k^m]$ at times $t_k, k \in [s, e]$, **the IMU propagation can be summarized** by

$$\text{Prop: } [\mathbf{x}_s, \mathbf{I}, \Sigma_s, \mathbf{z}_{s:e}] \rightarrow [\mathbf{x}_{e|s}, \boldsymbol{\Phi}(t_e, t_s), \Sigma_{e|s}]$$

Here, $\mathbf{x}_{e|s}$ is the predicted state at time t_e.

$\boldsymbol{\Phi}(t_e, t_s)$ is the state transition matrix from t_s to t_e.

We define the **error state for propagation**

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus^p \delta \mathbf{x}^p.$$

Σ_s and $\Sigma_{e|s}$ are the covariances of the error states $\delta \mathbf{x}_s^p$ and $\delta \mathbf{x}_{e|s}^p$ at t_s and t_e, respectively. Note we use the subscripts **s and e to denote the start and end times**.

Compute the transition matrix and covariance

Continuous system model

The continuous-time system model with control input $\mathbf{u}$ and Gaussian noise $\mathbf{n} \sim \mathbf{N}(0, \mathbf{Q}\delta(t))$ is

$$\frac{d}{dt}\mathbf{x} = f(\mathbf{x}, \mathbf{u}, \mathbf{n}),$$
$$\frac{d}{dt}\delta\mathbf{x} = \mathbf{F}(\mathbf{x}, \mathbf{u})\delta\mathbf{x} + \mathbf{G}(t)\,\mathbf{n}$$

Discretized system model

$$\mathbf{x}_k = g(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)$$
$$\delta\mathbf{x}_k = \boldsymbol{\Phi}(t_k, t_{k-1})\,\delta\mathbf{x}_{k-1} + \mathbf{B}_k\,\mathbf{w}_k$$
$$\boldsymbol{\Phi}(t_k, t_{k-1}) = \exp(\mathbf{F}(\mathbf{x}, \mathbf{u})\,dt) \approx \mathbf{I} + \mathbf{F}\,dt + 1/2\mathbf{F}^2\,dt^2$$
$$\boldsymbol{\Phi}(t_e, t_s) = \boldsymbol{\Phi}(t_e, t_{e-1}) \cdot \cdots \cdot \boldsymbol{\Phi}(t_{s+1}, t_s)$$
$$\Sigma_k = \boldsymbol{\Phi}(t_k, t_{k-1})\,\Sigma_{k-1}\,\boldsymbol{\Phi}^\top(t_k, t_{k-1}) + \int_{t_{k-1}}^{t_k} \boldsymbol{\Phi}\,(t_k, \tau)\mathbf{G}(\tau)\mathbf{Q}\mathbf{G}^\top(\tau)\,\boldsymbol{\Phi}^\top(t_k, \tau)\,d\tau$$

Preintegration Basics

The IMU preintegration operation can be summarized by

$$\text{Preint: } [\mathbf{b}_s, \mathbf{z}_{s:e}, \mathbf{q}] \rightarrow [\Delta\mathbf{x}_{se}, \mathbf{J}_{b_s}, \Sigma_{\Delta x}]$$

Here $\mathbf{b}_s$ is the IMU bias at t_s, and $\mathbf{q}$ is the IMU noise param vector.

Preintegrated inertial measurement

$$\Delta\mathbf{R} \doteq \Delta\mathbf{R}_{se} \doteq \prod_{k=s+1}^{e}\left(\int_{t_{k-1}}^{t_k} \omega\,(t)\,dt\right)$$
$$\Delta\mathbf{p} \doteq \Delta\mathbf{p}_e^s \doteq \int_{t_s}^{t_e}\int_{t_s}^{\tau} \mathbf{R}_t^s\,a(t)\,dt\,d\tau$$
$$\Delta\mathbf{v} \doteq \Delta\mathbf{v}_e^s \doteq \int_{t_s}^{t_e} \mathbf{R}_t^s\,a(t)\,dt$$
$$\Delta\mathrm{b} = \int_{t_s}^{t_e} n_b\ dt$$
$$\Delta t = t_e - t_s$$

Preintegrated bias Jacobian

$$\mathbf{J}_{b_s} = \frac{\partial\delta\mathbf{x}_{se}}{\partial\delta\mathbf{b}_s} \in \mathbb{R}^{15\times 6}$$

Preintegrated covariance

$$\Sigma_{\Delta\mathrm{x}} = \mathrm{cov}(\delta\mathbf{x}_{se}) \in \mathbb{R}^{15\times 15}$$

The preintegrated measurements are usually used as preintegrated inertial factors in optimization, *e.g.*,

$$\mathbf{r}_\theta = \mathrm{Log}\left(\mathbf{R}_{we}^\top\mathbf{R}_{w,e|s}\right)$$
$$\mathbf{r}_p = \mathbf{R}_{we}^\top\left(\mathbf{p}_{e|s} - \mathbf{p}_e\right)$$
$$\mathbf{r}_v = \mathbf{R}_{we}^\top\left(\mathbf{v}_{e|s} - \mathbf{v}_e\right)$$
$$\mathbf{r}_b = \mathbf{b}_{e|s} - \mathbf{b}_e$$

Preintegration from and to Propagation

Reuse propagation for preintegration

Given $\mathbf{x}_{e|s}, \boldsymbol{\Phi}(t_e, t_s), \Sigma_{e|s}$, **the preintegration measurement** $\Delta\mathbf{x}_{se}$ is obtained with the relations:

$$\Delta\mathbf{R} = \mathbf{R}_{ws}^\top\mathbf{R}_{w,e|s}$$
$$\Delta\mathbf{p} = \mathbf{R}_{ws}^\top\left[\mathbf{p}_{e|s} - \left(\mathbf{p}_s + \mathbf{v}_s\Delta t + \frac{1}{2}\mathbf{g}\Delta t^2\right)\right]$$
$$\Delta\mathbf{v} = \mathbf{R}_{ws}^\top(\mathbf{v}_{e|s} - \mathbf{v}_s - \mathbf{g}\Delta\mathrm{t})$$
$$\Delta\mathbf{b}_{se} = \mathbf{b}_{e|s} - \mathbf{b}_s$$

The preintegration Jacobian can be obtained by

$$\mathbf{J}_{b_s} = \frac{\partial\delta\mathbf{x}_{se}^g}{\partial\delta\mathbf{b}_s^g} = \left(\frac{\partial\delta\mathbf{x}_{e|s}^p}{\partial\delta\mathbf{x}_{se}^g}\right)^{-1}\frac{\partial\delta\mathbf{x}_{e|s}^p}{\partial\delta\mathbf{b}_s^p}\frac{\partial\delta\mathbf{b}_s^p}{\partial\delta\mathbf{b}_s^g}$$

Note $\frac{\partial\delta\mathbf{x}_{e|s}^p}{\partial\delta\mathbf{x}_{se}^g}$ follows from the definitions of $\delta\mathbf{x}_{e|s}^p$, $\delta\mathbf{x}_{se}^g$, and the above prediction equations. $\frac{\partial\delta\mathbf{b}_s^p}{\partial\delta\mathbf{b}_s^g}$ follows from the definitions of $\delta\mathbf{b}_s^p$ and $\delta\mathbf{b}_s^g$.

e.g., $\delta\mathbf{x}_{e|s}^p$ components can be defined by

$$\mathbf{p} = \hat{\mathbf{p}} + \delta\mathbf{p},\quad \mathbf{R} = \mathrm{Exp}(\delta\boldsymbol{\theta})\hat{\mathbf{R}}$$
$$\mathbf{v} = \hat{\mathbf{v}} + \delta\mathbf{v},\quad \mathbf{b}_g = \hat{\mathbf{b}}_g + \delta\mathbf{b}_g$$
$$\mathbf{b}_a = \hat{\mathbf{b}}_a + \delta\mathbf{b}_a$$

e.g., $\delta\mathbf{x}_{se}^g$ components can be defined by

$$\mathbf{R} = \hat{\mathbf{R}}\mathrm{Exp}(\delta\boldsymbol{\theta}),\quad \mathbf{p} = \hat{\mathbf{p}} + \mathbf{R}\delta\mathbf{p}$$
$$\mathbf{v} = \hat{\mathbf{v}} + \mathbf{R}\delta\mathbf{v},\quad \mathbf{b}_a = \hat{\mathbf{b}}_a + \delta\mathbf{b}_a$$
$$\mathbf{b}_g = \hat{\mathbf{b}}_g + \delta\mathbf{b}_g$$

The preintegration covariance can be computed from the propagated covariance $\Sigma_{e|s} = \mathrm{cov}\left(\delta\mathbf{x}_{e|s}^p\right)$

$$\Sigma_{\Delta\mathrm{x}} = \left(\frac{\partial\delta\mathbf{x}_{e|s}^p}{\partial\delta\mathbf{x}_{se}^g}\right)^{-1}\mathrm{cov}\left(\delta\mathbf{x}_{e|s}^p\right)\left(\frac{\partial\delta\mathbf{x}_{e|s}^p}{\partial\delta\mathbf{x}_{se}^g}\right)^{-\top}$$

$\mathrm{cov}\left(\delta\mathbf{x}_{e|s}^p\right)$ is obtained by propagation starting with zero covariance.

Reuse preintegration for propagation

Given $\Delta\mathbf{x}_{se}$, $\mathbf{J}_{b_s}, \Sigma_{\Delta\mathrm{x}}$, **the propagated state** is obtained by

$$\mathbf{R}_{w,e|s} \doteq \mathbf{R}_{ws}\Delta\mathbf{R}$$
$$\mathbf{p}_{e|s} \doteq \mathbf{p}_s + \mathbf{v}_s\Delta t + \frac{1}{2}\mathbf{g}\Delta t^2 + \mathbf{R}_{ws}\Delta\mathbf{p}$$
$$\mathbf{v}_{e|s} \doteq \mathbf{v}_s + \mathbf{R}_{ws}\Delta\mathbf{v} + \mathbf{g}\Delta t$$
$$\mathbf{b}_{e|s} \doteq \mathbf{b}_s + \Delta\mathbf{b}_{se}$$

The transition matrix is obtained by taking the differential of the above equations denoted by $\mathbf{f}$.

$$\delta\mathbf{x}_{e|s}^p = \boldsymbol{\Phi}_{e|s}\delta\mathbf{x}_s^p + \mathbf{G}_{e|s}\delta\mathbf{x}_{se}^g$$
$$\mathbf{G}_{e|s} = \frac{\partial\delta\mathbf{f}}{\partial\delta\mathbf{x}_{se}^g} = \begin{bmatrix}\mathbf{G}_{e|s}^{nn} & \mathbf{0}_{9\times 6}\\ \mathbf{0}_{6\times 9} & \mathbf{I}_6\end{bmatrix}$$
$$\boldsymbol{\Phi}_{e|s} = \frac{\partial\delta\mathbf{f}}{\partial\delta\mathbf{x}_s^p} + \begin{bmatrix}\mathbf{0} & \mathbf{G}_{e|s}\frac{\partial\delta\mathbf{x}_{se}^g}{\partial\delta\mathbf{b}_s^g}\frac{\partial\delta\mathbf{b}_s^g}{\partial\delta\mathbf{b}_s^p}\end{bmatrix}$$
$$= \frac{\partial\delta\mathbf{f}}{\partial\delta\mathbf{x}_s^p} + \begin{bmatrix}\mathbf{0} & \mathbf{G}_{e|s}^{nn}\mathbf{J}_{b_s}^n\mathbf{P}_b\\ \mathbf{0} & \mathbf{0}_{6\times 6}\end{bmatrix}$$
$$\mathbf{P}_b = \frac{\partial\delta\mathbf{b}_s^g}{\partial\delta\mathbf{b}_s^p},\qquad \frac{\partial\delta\mathbf{x}_{se}^g}{\partial\delta\mathbf{b}_s^g} = \mathbf{J}_{b_s} = \begin{bmatrix}\mathbf{J}_{b_s}^n\\ \mathbf{0}_{6\times 6}\end{bmatrix}$$

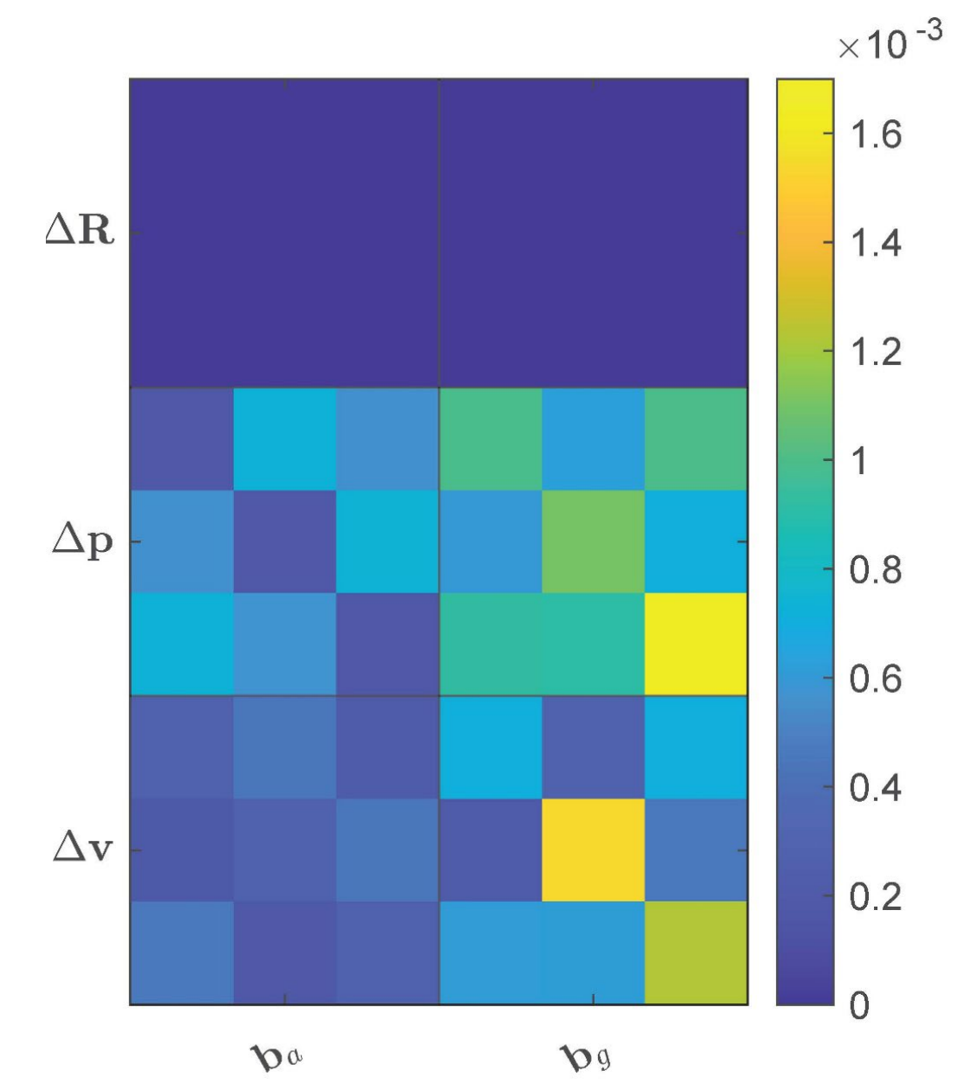
The propagated covariance is computed by

$$\Sigma_{e|s} = \boldsymbol{\Phi}_{e|s}\Sigma_s\boldsymbol{\Phi}_{e|s}^\top + \mathbf{G}_{e|s}\Sigma_{\Delta\mathrm{x}}\mathbf{G}_{e|s}^\top$$

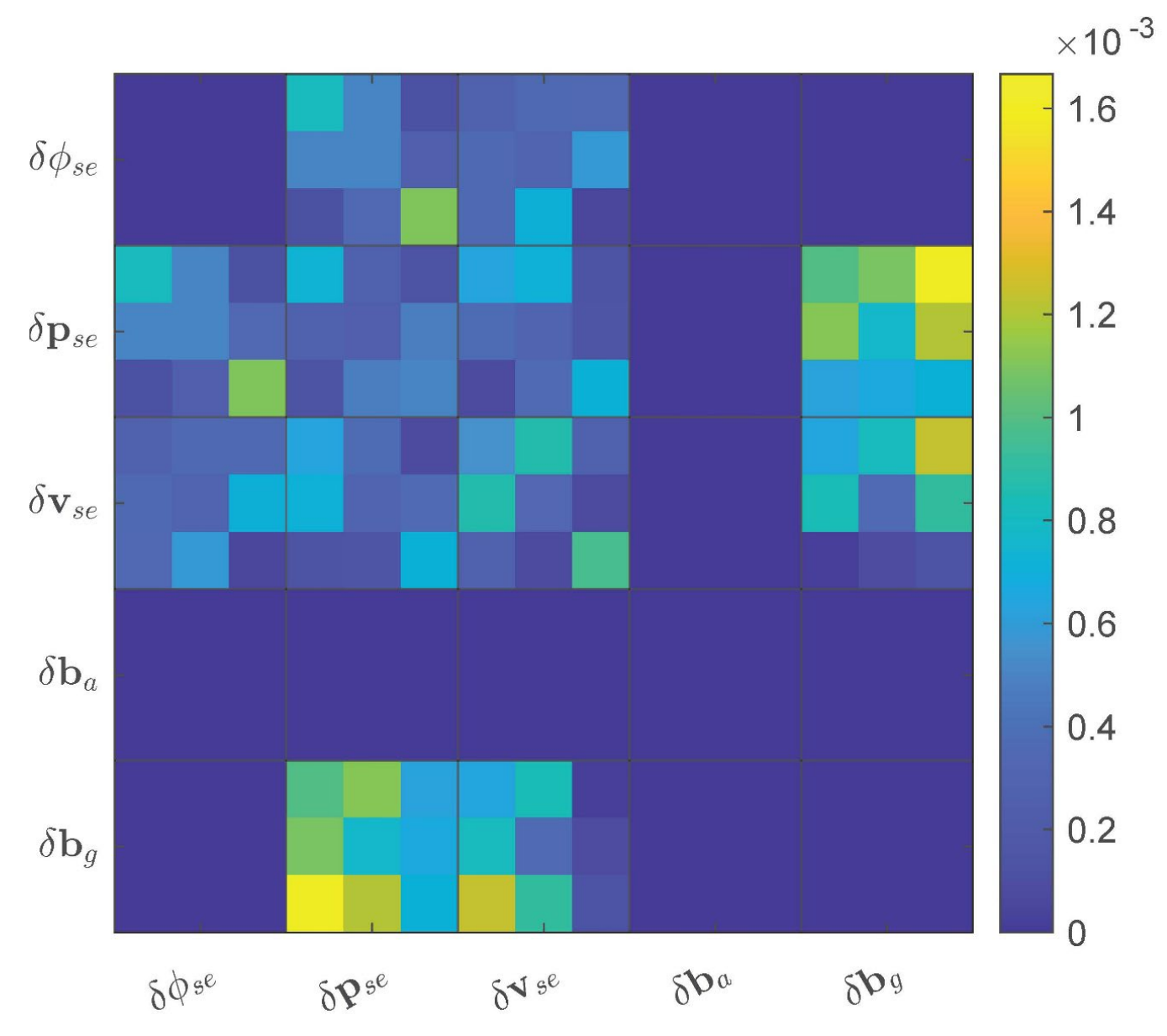
with the preintegrated measurement covariance $\Sigma_{\Delta\mathrm{x}}$ and the covariance of the initial state Σ_s.

Preintegration from Propagation Test

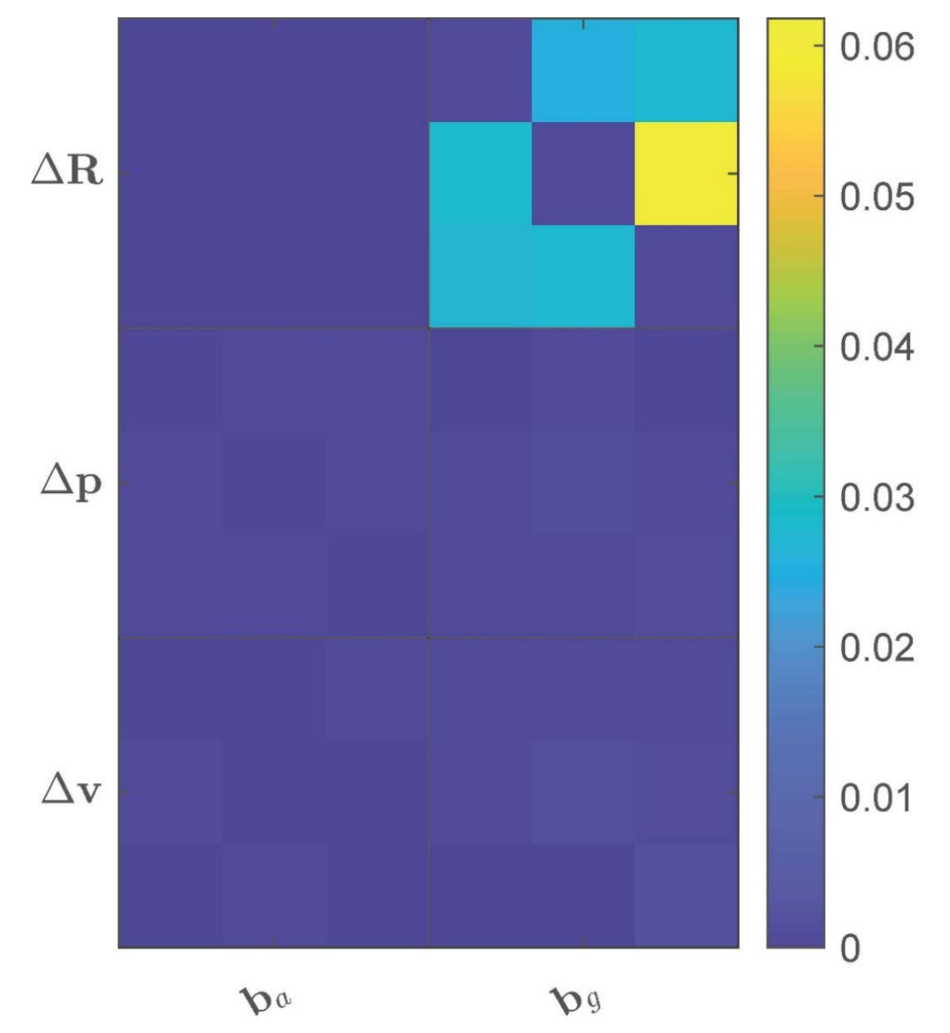
RK4 Propagation + Conversion Versus GTSAM Preintegration



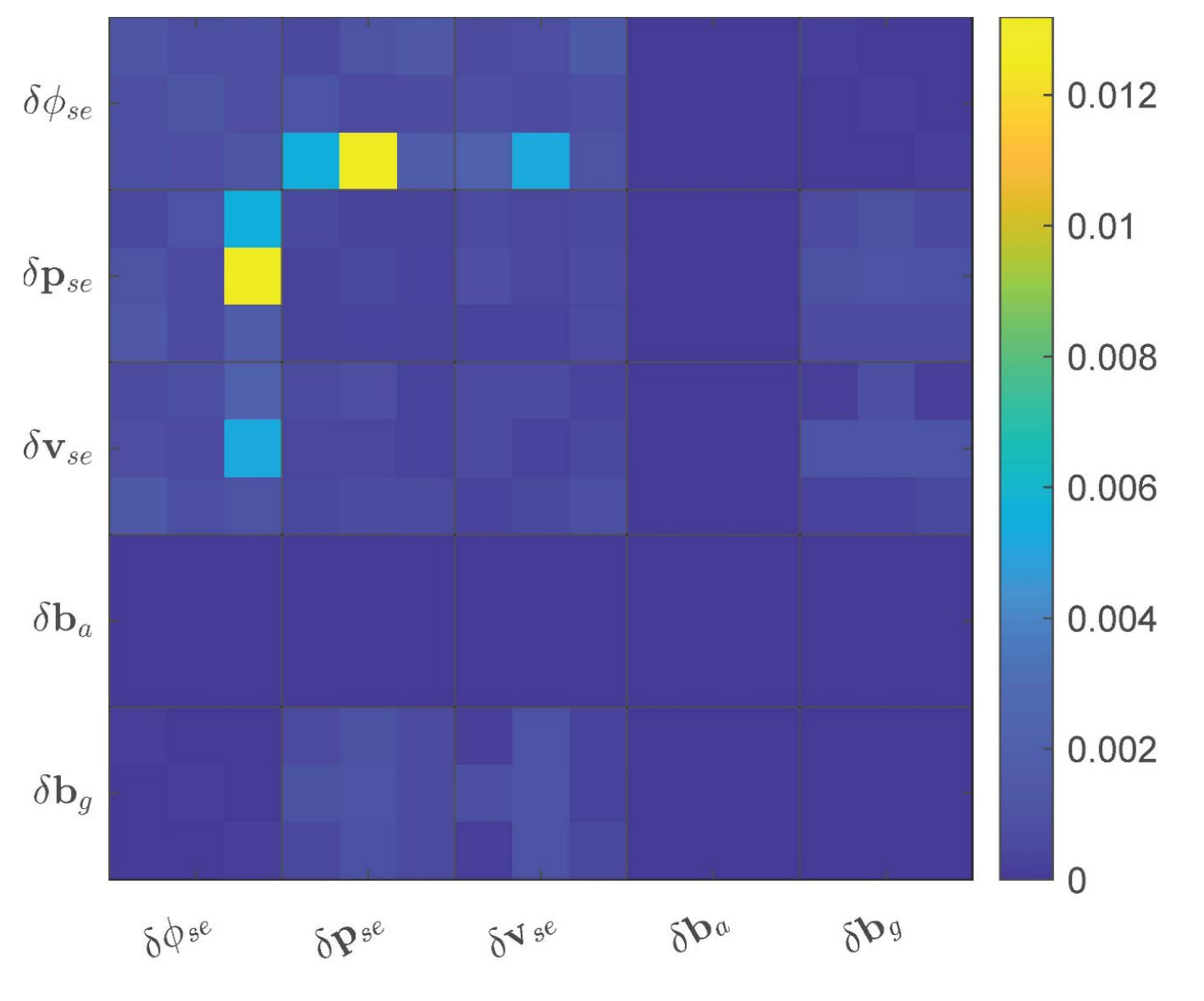
Preintegration bias Jacobian difference with GTSAM manifold preintegration



Preintegration covariance difference with GTSAM manifold preintegration



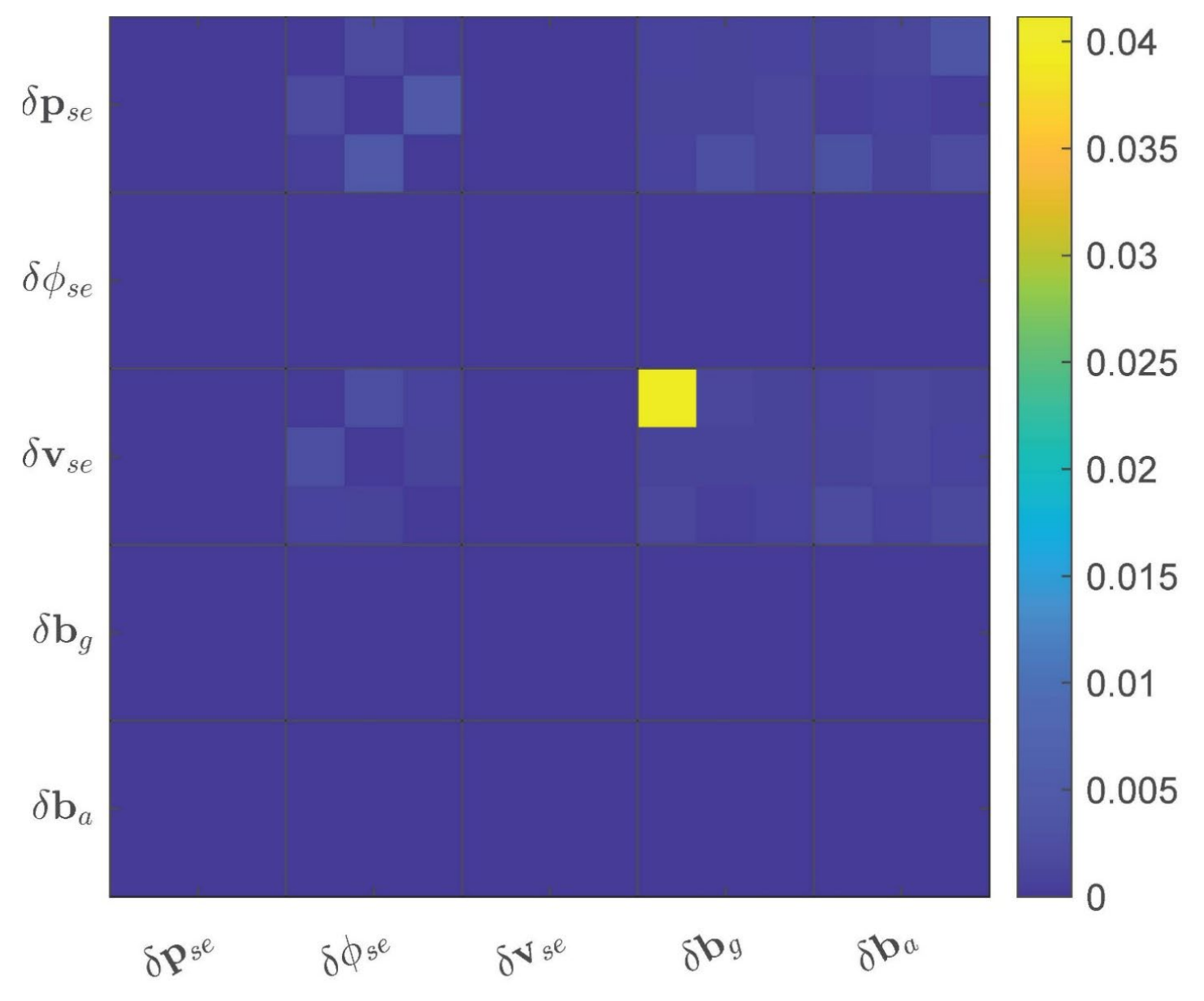
Preintegration bias Jacobian difference with GTSAM tangent preintegration



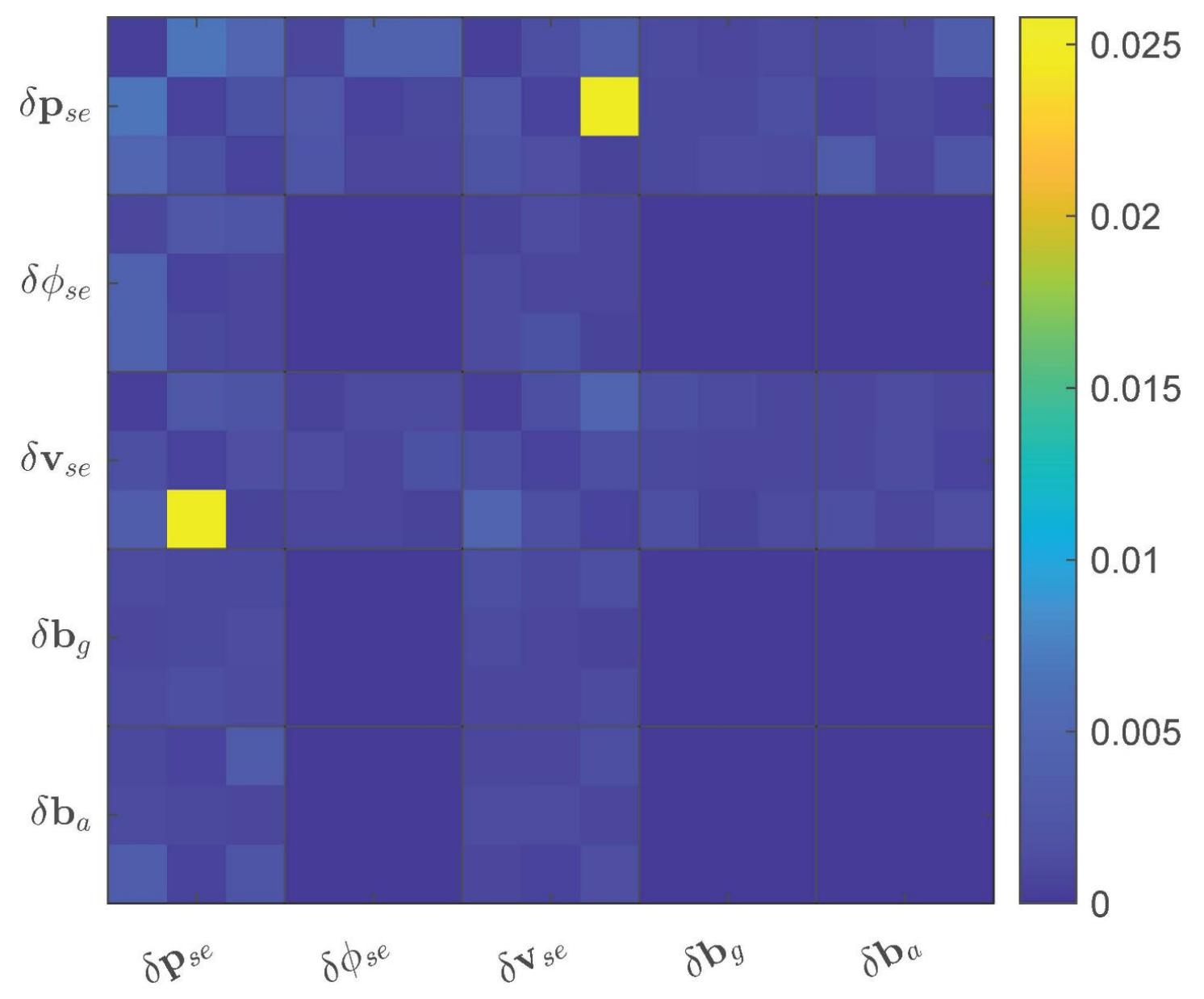
Preintegration covariance difference with GTSAM tangent preintegration

Propagation from Preintegration Test

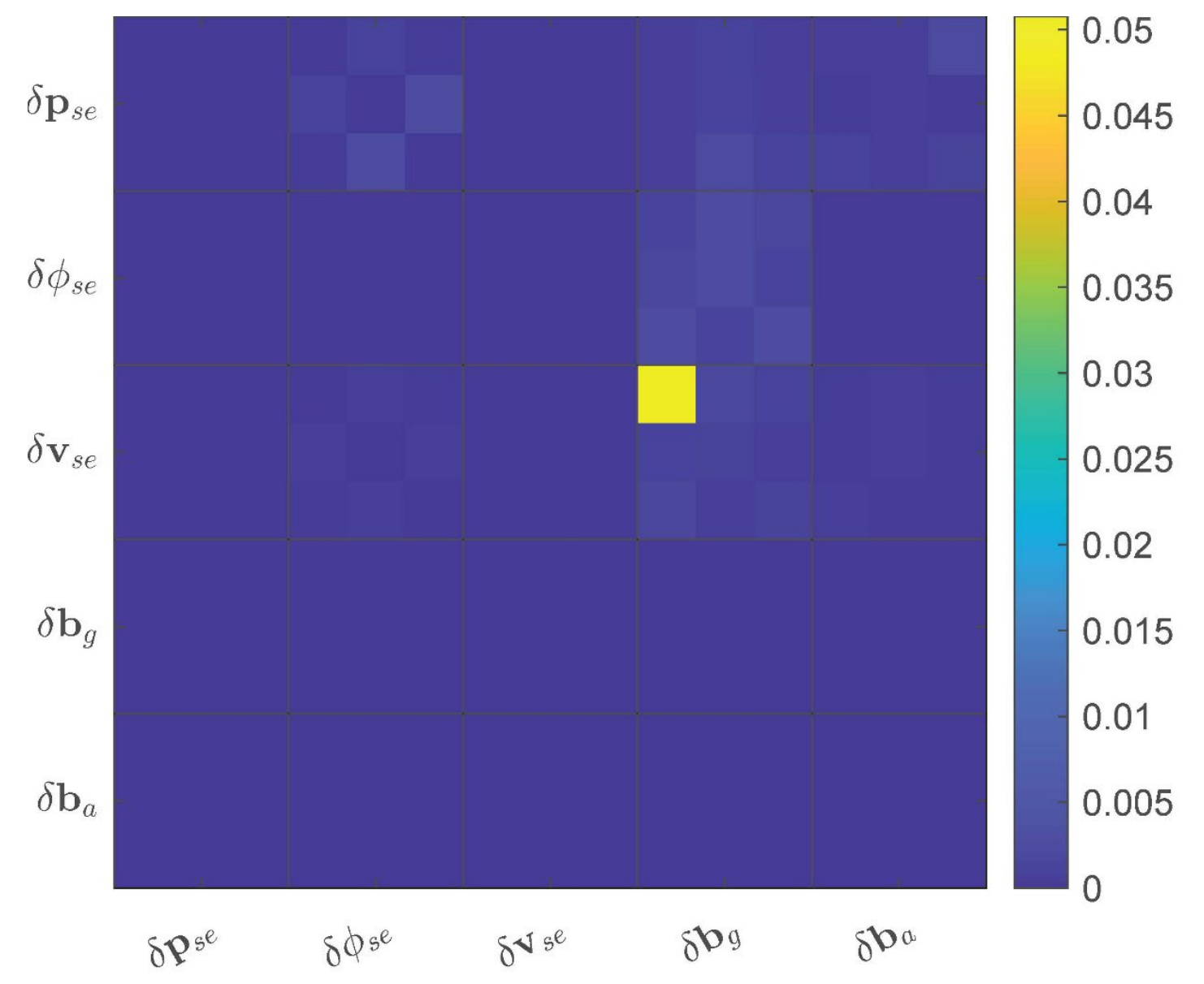
RK4 Propagation Versus GTSAM Preintegration + Conversion



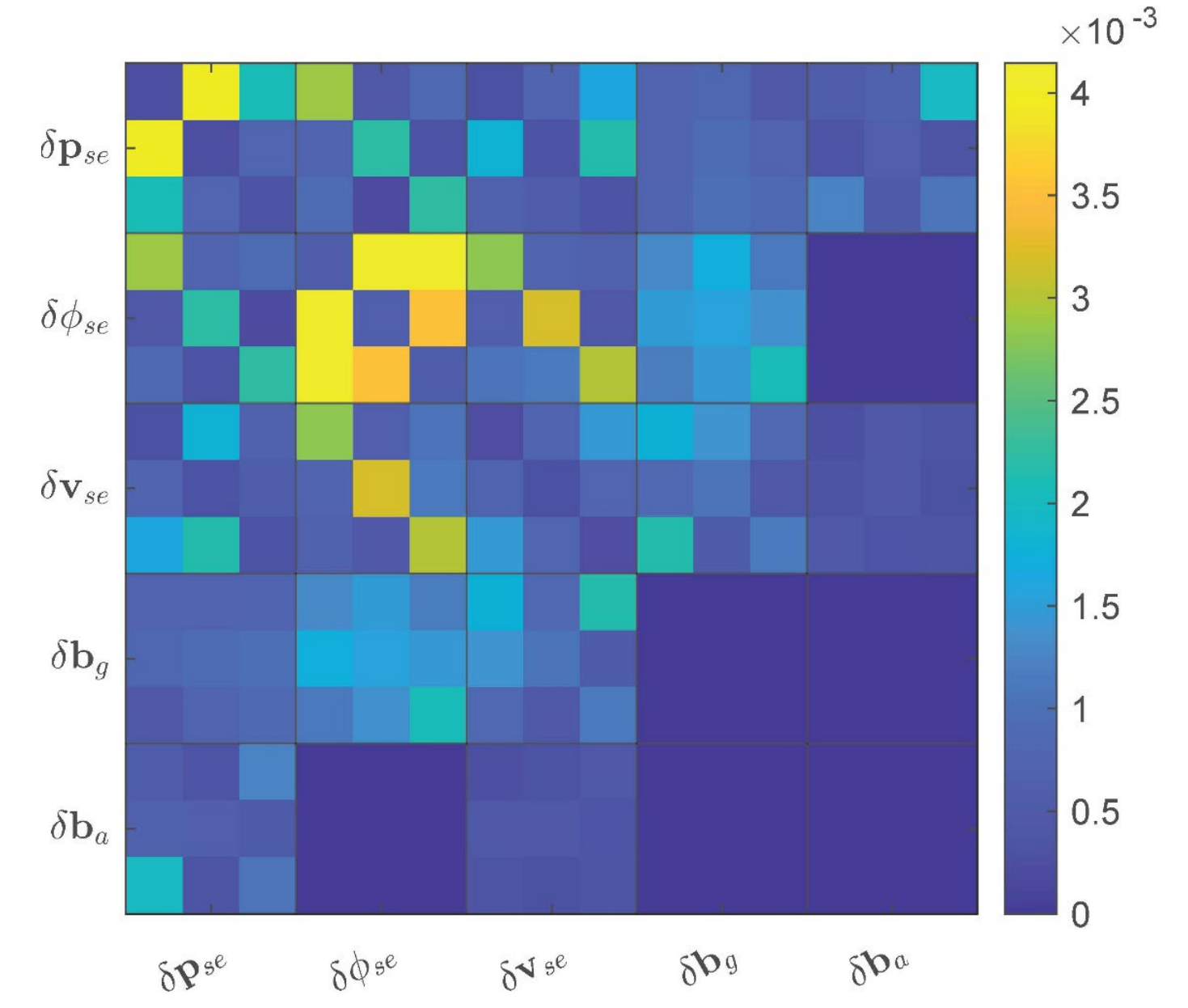
Transition matrix difference with GTSAM manifold preintegration



Propagation covariance difference with GTSAM manifold preintegration



Transition matrix difference with GTSAM tangent preintegration



Propagation covariance difference with GTSAM tangent preintegration

Conclusions

- IMU preintegration can be achieved by wrapping a classic IMU propagation module, and a preintegration module can in turn be used to recover propagation quantities.
- This perspective simplifies the reuse of existing propagation code, supports translation across different error-state definitions, and provides practical consistency checks for preintegration implementations.

References

[1] J. Huai, IMU Propagation as Preintegration. ISPRS 2026, Toronto, Canada. https://arxiv.org/abs/2605.28279